

# On the Implicit Ontological Transitions in Classical Analysis

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## **Abstract**

Classical real analysis is extraordinarily successful as a computational framework. However, its standard formalism contains implicit transitions between different mathematical spaces that are not explicitly represented in notation. In this paper we argue that the limit operator is not merely a numerical operation but an operator inducing a change of descriptive level. We contrast this situation with the explicit ontological extension performed in complex analysis and review historical and modern critiques of infinitesimal and limit-based reasoning. Our aim is not to reject analysis, but to clarify the scope and interpretation of its core operators.

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# 1 Introduction

Classical analysis has long served as the foundational language of mathematical physics and applied mathematics. Despite its effectiveness, concerns regarding the conceptual status of infinitesimals, limits, and continuity have accompanied it since its origin. These concerns are often dismissed as philosophical; here we argue that they can be formulated as precise questions about the structure of the formalism itself.

## 2 Historical Background

### 2.1 Newton and Fluxions

Newton introduced fluxions as quantities generated by motion, while simultaneously expressing reservations about their logical status [7]. His geometric reformulations can be seen as attempts to avoid explicit appeal to infinitesimals.

### 2.2 Leibniz and Infinitesimals

Leibniz treated infinitesimals as *fictiones utiles*, a position that acknowledged their instrumental rather than ontological character [6].

## 3 Historical Critiques of Infinitesimals and Continuity

Concerns regarding infinitesimals, limits, and the continuum are not a recent development. Since the inception of calculus, a number of mathematicians and philosophers have raised objections to the logical, operational, and interpretive status of its core concepts. The present analysis situates itself within this historical line of critique, while differing in emphasis and scope.

### 3.1 George Berkeley (18th Century)

One of the earliest and most explicit critiques of the Newton–Leibniz calculus was articulated by George Berkeley in *The Analyst* [1]. Berkeley famously referred to infinitesimals as “ghosts of departed quantities”, arguing that they were treated inconsistently within derivations: assumed to be nonzero during intermediate steps and later identified with zero.

Berkeley’s critique targeted a logical asymmetry rather than a numerical error. However, as Berkeley was not himself a mathematician, his objections were largely dismissed or bypassed rather than formally addressed. The modern reformulation of calculus replaced infinitesimals with limits, but preserved the underlying structure that Berkeley had identified.

The present work revisits Berkeley’s argument from a technical standpoint, focusing not on logical rhetoric but on computability and structural transitions induced by the limit operator.

### 3.2 Leopold Kronecker

Leopold Kronecker is well known for his dictum:

“God created the integers; all else is the work of man.”

Kronecker rejected the notion of actual infinity and denied any ontological status to the continuum. He regarded only finite, constructive procedures as mathematically legitimate [5].

Kronecker’s position anticipates the present emphasis on stepwise, algorithmic operations. However, his critique was directed primarily at the existence of infinite objects, whereas the

present analysis focuses on the implicit transitions between operational and classificatory levels within accepted formalisms.

### **3.3 Luitzen Egbertus Jan Brouwer**

Brouwer's intuitionism constituted a foundational challenge to classical analysis by rejecting the law of excluded middle and the existence of mathematical objects without explicit construction. In Brouwer's view, the continuum is not a completed entity but an ongoing mental construction [3].

While sharing the rejection of completed infinities, the present approach differs significantly from Brouwer's. Intuitionism grounds mathematical existence in subjective acts of construction, whereas the present framework emphasizes objective discreteness of events and procedures, independent of mental activity.

### **3.4 Errett Bishop**

Bishop's program of constructive analysis sought to reconstruct large portions of classical analysis using only algorithmically meaningful definitions and proofs [2]. In doing so, Bishop demonstrated that much of analysis could be retained without appeal to non-constructive existence proofs or purely abstract limits.

This work aligns with Bishop in recognizing the operational inadequacy of classical limits. However, whereas Bishop aimed to repair analysis from within by enforcing constructivity, the present analysis questions the ontological interpretation of limits rather than their computational utility.

### **3.5 Hermann Weyl**

Hermann Weyl expressed sustained concern about the logical status of the continuum and acknowledged that physical theories do not require genuine continuity at a fundamental level [9]. At the same time, Weyl ultimately retained the continuum for reasons of mathematical elegance and expressive power.

The present work departs from Weyl precisely at this point. Rather than preserving continuity for aesthetic or unificatory reasons, it seeks to make explicit the structural assumptions underlying its use.

### **3.6 Thoralf Skolem**

Skolem's work, particularly the Löwenheim–Skolem theorem, revealed that formal systems capable of expressing uncountable structures admit countable models. This result undermines naive interpretations of set-theoretic size and suggests that the continuum does not possess an invariant formal representation [8].

Skolem's contribution represents an internal critique of the continuum within logic. The present analysis complements this by examining how the continuum is reintroduced into applications through implicit identifications rather than formal necessity.

### **3.7 Richard Feynman**

Although not a foundational mathematician, Richard Feynman repeatedly expressed pragmatic skepticism toward the conceptual foundations of analysis. He described calculus as a tool that "works, but we do not really know why", and consistently emphasized procedural, discrete computation in practice [4].

Feynman's perspective highlights a divergence between formal analytical language and actual computational practice. The present work can be seen as a theoretical articulation of this gap, focusing on the role of limits as classificatory rather than generative operations.

## 4 Comparison with Complex Analysis

The introduction of the imaginary unit  $i = \sqrt{-1}$  explicitly extends the real numbers to the complex plane. This extension:

- introduces a new dimension (phase),
- modifies algebraic rules,
- clearly distinguishes  $\mathbb{R}$  from  $\mathbb{C}$ .

No analogous explicit extension accompanies the use of limits in real analysis, despite a comparable change in descriptive level.

Having established a paradigm of explicit structural extension in complex analysis, we now turn to situations in real analysis where comparable transitions occur without explicit notation.

## 5 Examples of Implicit Transitions

### 5.1 Arithmetic with Limits

Expressions such as

$$\left( \lim_{n \rightarrow \infty} \frac{1}{n} \right) + 1$$

treat the result of an evaluative operator as an element of the original parameter space, thereby suppressing the distinction between spaces.

### 5.2 Physical Interpretation

In physical applications, quantities derived via limits are often treated as directly measurable states, despite originating from procedures that classify behavior rather than represent physical events.

## 6 Relation to Constructive and Intuitionistic Approaches

Constructive mathematics and intuitionism, notably in the work of Brouwer and Bishop [2], address related concerns by restricting acceptable operations to those with explicit constructions. Our analysis differs in focusing on the explicit representation of operator-induced space changes.

## 7 Limits as Operators

**Definition 1.** *Let  $(x_n)$  be a sequence in a parameter space  $P$ . The limit operator  $\lim$  is an external operator that maps descriptions of  $(x_n)$  into a value in a possibly different evaluative space  $C$ .*

**Proposition 1.** *The value produced by the limit operator does not, in general, belong to the parameter space  $P$  of the original sequence.*

*Proof.* For the sequence  $x_n = 1/n$ , all elements satisfy  $x_n > 0$ . The value 0 does not appear in the sequence and is introduced only after applying the limit operator.  $\square$

## 8 Limit as a Classification Operator

In standard presentations of real analysis, the limit is introduced as a numerical notion characterizing the behavior of a sequence or function. However, a closer inspection shows that the limit operator does not act within the same space as the objects it is applied to. Instead, it performs a classification of asymptotic behavior.

**Definition 2.** Let  $(x_n)$  be a sequence defined on a parameter space  $P$ . The limit operator

$$\lim_{n \rightarrow \infty}$$

is an external operator that maps the description of  $(x_n)$  into a classification space  $C$ , whose elements represent asymptotic equivalence classes rather than attainable states of the sequence.

In this sense, the value produced by the limit operator is not the result of a completed process, but the outcome of an evaluative criterion applied to an infinite description.

**Proposition 2.** The limit of a sequence, when it exists, represents a classification of the sequence's asymptotic behavior and not an element generated by the sequence itself.

*Proof.* Consider the sequence  $x_n = 1/n$ . For all finite  $n$ ,  $x_n > 0$ , and no element of the sequence equals 0. The value 0 arises only after the application of the limit operator, indicating that it is introduced by the classification procedure rather than produced by the sequence.  $\square$

From this perspective, the expression

$$\lim_{n \rightarrow \infty} x_n = L$$

should be read as:

“The sequence  $(x_n)$  belongs to the asymptotic class characterized by  $L$  under the chosen convergence criterion.”

This interpretation aligns the limit operator with other classification procedures commonly used in mathematics and physics, such as stability analysis or convergence tests, where the output describes qualitative behavior rather than a physical or computationally reachable state.

**Remark 1.** When the output of the limit operator is subsequently treated as an element of the original parameter space  $P$ , an implicit identification between classification space and parameter space occurs. This identification is not enforced by the formal definition of the limit itself and constitutes an additional interpretive step.

The absence of an explicit notation for this space transition contrasts with the introduction of complex numbers, where the extension from  $\mathbb{R}$  to  $\mathbb{C}$  is made explicit and accompanied by revised algebraic rules. The limit operator, by comparison, induces a change of descriptive level without a corresponding syntactic marker.

### 8.1 Limits and Computability

It is sometimes argued that the limit operator should be regarded as a functional acting on sequences or functions. However, this characterization obscures a critical structural distinction.

A functional is an operator that preserves computability in the following sense: given a computable description of its input, the functional produces a computable description of its output. In contrast, the limit operator generally does not preserve computability.

**Remark 2.** Given a computable sequence  $(x_n)$ , the value of  $\lim_{n \rightarrow \infty} x_n$ , when it exists, need not be computable. Moreover, no finite procedure can, in general, determine whether a given computable sequence converges, nor identify its limit with arbitrary precision.

From this perspective, the limit operator differs fundamentally from standard functionals used in analysis. Rather than extending computation, it terminates it by collapsing an infinite description into a classificatory outcome.

This property further supports the interpretation of the limit as an external classification operator rather than as an internal numerical operation. The output of the limit operator encodes asymptotic behavior, but does not constitute a value generated by a computable transformation of the input.

## 9 Arithmetic on Classified Values

Once the limit operator is interpreted as a classification operator, a natural question arises concerning the admissibility of arithmetic operations involving its output. In standard practice, classified values are routinely combined with elements of the original parameter space. In this section we examine the formal status of such operations.

**Definition 3.** Let  $P$  denote a parameter space and  $C$  a classification space induced by an external operator such as  $\lim$ . An arithmetic operation is said to be internally valid if it maps elements of a single space into that same space.

Arithmetic operations such as addition and multiplication are internally valid on  $P$ , as well as on  $C$ , provided that both operands belong to the same space. However, no internal rule of arithmetic specifies how elements of  $C$  should be combined with elements of  $P$ .

**Proposition 3.** Arithmetic expressions combining classified values with parameter values implicitly assume an identification between classification space  $C$  and parameter space  $P$ .

*Proof.* Consider the expression

$$\left( \lim_{n \rightarrow \infty} \frac{1}{n} \right) + 1.$$

The term  $\lim_{n \rightarrow \infty} \frac{1}{n}$  belongs to the classification space  $C$ , while 1 is an element of the parameter space  $P$ . The addition operator is defined only within a single space; therefore, this expression presupposes an identification  $C \equiv P$ . Such an identification is not enforced by the definition of the limit operator itself and constitutes an additional assumption.  $\square$

This observation does not imply that such expressions are computationally invalid. Rather, it highlights that their validity depends on a convention that suppresses the distinction between spaces.

**Remark 3.** A closely related situation occurs in complex analysis. When extending  $\mathbb{R}$  to  $\mathbb{C}$ , arithmetic between real and complex numbers is permitted only after explicitly embedding  $\mathbb{R}$  into  $\mathbb{C}$ . No analogous embedding is specified when classified values produced by limits are reintroduced into parameter-level arithmetic.

From a structural perspective, arithmetic on classified values can be made well-defined by introducing an explicit projection or embedding operator

$$\pi : C \rightarrow P,$$

which maps classification outcomes back into the parameter space. Absent such an operator, expressions combining elements of  $C$  and  $P$  remain syntactically correct but semantically undetermined.

**Remark 4.** *In applied contexts, this implicit projection is often justified by empirical stability or numerical convergence. While sufficient for computation, such justifications operate at the level of practice rather than formal structure.*

The distinction emphasized here parallels similar concerns in type theory and dimensional analysis, where operations between quantities of different types require explicit coercions. Making such transitions explicit in real analysis would preserve computational utility while improving formal clarity.

## 10 Explicit Embeddings and Phase Markers

The analysis of arithmetic on classified values suggests that, whenever outputs of an external operator are reused within parameter-level expressions, an explicit embedding or projection should be specified. This section proposes a minimal formal device to represent such transitions.

### 10.1 Explicit Embeddings

Let  $P$  denote a parameter space and  $C$  a classification space induced by an operator such as  $\lim$ . To combine elements of  $C$  with elements of  $P$  in a well-defined manner, an explicit embedding

$$\iota : C \rightarrow \tilde{P}$$

must be introduced, where  $\tilde{P}$  denotes an extended parameter space.

In the absence of such an embedding, arithmetic expressions mixing elements of  $C$  and  $P$  implicitly assume  $\tilde{P} \equiv P$ , thereby suppressing the transition between descriptive levels.

### 10.2 Phase Markers

An alternative to explicit embeddings is to retain the distinction between spaces by augmenting classified values with a formal marker indicating their origin. We denote such an augmented value by

$$L^\varphi,$$

where  $L \in \mathbb{R}$  represents the numerical projection and  $\varphi$  denotes a phase marker identifying the value as the outcome of a classification procedure.

For example, the statement

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

may be represented as

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0^\varphi,$$

where  $\varphi$  encodes the fact that the value 0 arises from asymptotic classification rather than from the parameter space of the sequence.

### 10.3 Relation to Complex Analysis

This construction parallels the extension from  $\mathbb{R}$  to  $\mathbb{C}$ , where a real number  $x$  is embedded as  $x + 0i$ . The imaginary component serves as a structural marker indicating membership in the complex plane. Similarly, the phase marker  $\varphi$  signals a change in descriptive level without altering the numerical projection.

Unlike the complex case, where the additional dimension enables new forms of internal arithmetic, the phase marker introduced here serves solely as a bookkeeping device. Its purpose is not to enrich the numerical structure, but to prevent unintended identifications between classified values and parameter-level states.

## 10.4 Operational Consequences

When phase markers are retained, arithmetic expressions involving classified values require explicit rules governing their interaction with elements of the parameter space. Alternatively, phase markers may be eliminated via a projection

$$\pi : \tilde{P} \rightarrow P,$$

which must be specified as part of the formalism.

In applied contexts, such projections are often justified pragmatically by numerical convergence or empirical robustness. The present framework does not challenge such practices, but clarifies that their legitimacy rests on additional assumptions rather than on the intrinsic properties of the limit operator.

## 11 Implications for Mathematical Practice

The preceding analysis does not challenge the computational validity of classical real analysis. Instead, it has implications for how its core operators are interpreted, documented, and employed within mathematical practice.

### 11.1 Explicit Typing of Operators

Interpreting the limit as a classification operator suggests that it should be treated as a typed operation whose output does not automatically belong to the same space as its input. While informal usage often suppresses this distinction, making the typing explicit would align real analysis with established practices in areas such as type theory and dimensional analysis.

This explicit typing would not alter standard results, but would clarify which steps rely on additional identifications or projections.

### 11.2 Separation of Computation and Interpretation

In many applications, limits are used as intermediate tools in computations, after which their results are interpreted as numerical values. The present framework distinguishes these two stages:

1. computation within a parameterized or functional description,
2. interpretation of the classified outcome as a usable numerical quantity.

Recognizing this separation helps prevent the conflation of evaluative criteria with attainable states, without restricting practical usage.

### 11.3 Relation to Existing Formalisms

Several existing approaches implicitly address related concerns. Constructive analysis, intuitionistic mathematics, and domain theory all impose restrictions or structural annotations on admissible operations. The perspective proposed here differs in that it does not restrict the use of limits, but emphasizes the explicit representation of their structural role.

In this sense, the proposal is orthogonal to foundational programs that seek to replace classical analysis. It aims instead at improving the transparency of standard practice.

## 11.4 Pedagogical Consequences

From a pedagogical standpoint, introducing limits as classification tools rather than as completed processes may reduce conceptual ambiguities commonly encountered by students. Explicitly distinguishing between sequences, their asymptotic classes, and the numerical representatives of those classes provides a clearer account of why certain operations are permitted and others require justification.

Such an approach may also help clarify the relationship between real analysis and numerical methods, where finite procedures and convergence criteria are already treated as distinct notions.

## 11.5 Scope and Limitations

The considerations presented here do not imply that standard analytical results are incorrect, nor that existing notation must be abandoned. They indicate, rather, that classical analysis relies on conventions that are usually left implicit.

Making these conventions explicit is a matter of formal clarity rather than foundational revision.

# 12 Implications for Physical Modeling

The distinction between parameter spaces and classification spaces has direct implications for physical modeling, particularly in contexts where analytical results are interpreted as physical states or processes.

## 12.1 Limits and Physical States

In many physical models, quantities obtained via limits are treated as physically attainable states. From the perspective developed in this paper, such quantities originate as outputs of classification procedures rather than as elements generated by the underlying physical process.

This observation does not invalidate the use of limits in modeling, but it clarifies that additional interpretive steps are required when classified values are identified with physical states. Such identifications function as modeling assumptions rather than as consequences of the formalism itself.

## 12.2 Continuous Models and Discrete Measurement

Physical measurements are inherently discrete, finite, and bounded by instrumental resolution. Continuous models, by contrast, employ limits to describe idealized behavior across unbounded scales.

Interpreting limits as classification operators aligns continuous modeling with this operational reality: the limit characterizes a regime of behavior rather than a directly observable quantity. This interpretation helps reconcile the effectiveness of continuous models with the discreteness of empirical data.

## 12.3 Stability, Regimes, and Asymptotic Classes

In practical physics, limits are often used to identify stable regimes, equilibrium states, or asymptotic behaviors. Viewed as classification outcomes, such limits naturally correspond to equivalence classes of behavior rather than to precise dynamical endpoints.

This perspective emphasizes that many physically meaningful conclusions concern qualitative stability and robustness rather than exact numerical values. The success of limit-based

models thus reflects their ability to classify behavioral regimes rather than to describe detailed micro-dynamics.

## 12.4 Relation to Relativistic and Quantum Models

In relativistic and quantum theories, formal entities such as spacetime metrics or quantum states are frequently treated as primary physical objects. At the same time, observable content is extracted through measurement procedures yielding finite outcomes.

The present framework highlights that these formalisms operate at a level where classified structures organize possible observations. Recognizing this role does not diminish their predictive power, but it clarifies the distinction between model-internal constructs and observable events.

## 12.5 Model Validity and Interpretive Discipline

The effectiveness of physical models depends less on the ontological status of their internal constructs than on the stability of the classifications they induce. Explicitly acknowledging classification steps can therefore improve interpretive discipline without altering computational practice.

Such clarity is particularly relevant in domains where multiple models yield equivalent predictions while employing different internal representations. In these cases, limits function as organizing tools rather than as direct descriptions of physical processes.

# 13 Conclusion

This paper has examined the role of the limit operator in classical real analysis from a structural perspective. Rather than treating limits as numerical endpoints or completed processes, we have argued that they function as classification operators, mapping parameterized descriptions into asymptotic equivalence classes.

By distinguishing between parameter spaces and classification spaces, we have shown that common analytical expressions implicitly rely on additional identifications when classified values are reintroduced into arithmetic. These identifications are not mandated by the formal definition of limits and therefore constitute conventions of practice rather than intrinsic properties of the formalism.

A comparison with complex analysis illustrates that mathematics already possesses well-established mechanisms for handling extensions of numerical structure through explicit embeddings and revised operational rules. Applying similar discipline to real analysis would not alter its computational power, but would improve the transparency of its core operations.

The proposed use of explicit embeddings or phase markers is intended as a minimal formal clarification rather than a foundational revision. It preserves standard analytical results while making structural assumptions visible. In this sense, the present work does not seek to replace classical analysis, but to articulate a clearer account of how its operators function within mathematical and physical modeling.

Recognizing limits as classificatory tools provides a coherent explanation for the effectiveness of continuous models in contexts characterized by discrete measurement and finite computation. It also offers a framework for separating computational success from interpretive assumptions, thereby contributing to greater conceptual discipline across mathematical practice and its applications.

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